The Truth about Königsberg

Brian Hopkins and Robin J. Wilson

Brian Hopkins (bhopkins@spc.edu) is an assistant professor at St. Peter's College, a Jesuit liberal arts college in Jersey City, New Jersey. He received his Ph.D. from the University of Washington for work on algebraic combinatorics pertinent to the representation theory of Lie algebras. Other professional interests include graph theory and math education. He also enjoys choral singing, poetry, and New York City.

Robin Wilson (r.j.wilson@open.ac.uk) is Head of the Pure Mathematics Department at the Open University, and a Fellow of Keble College, Oxford University. He has written and edited about two dozen books on subjects ranging from graph theory to the history of mathematics, and has recently written a book on the history and proof of the four-color problem. He is very interested in music, and he has just co-edited a book on music and mathematics.

Euler's 1736 paper on the bridges of Königsberg is widely regarded as the earliest contribution to graph theory—yet Euler's solution made no mention of graphs. In this paper we place Euler's views on the Königsberg bridges problem in their historical context, present his method of solution, and trace the development of the present-day solution.

What Euler didn't do

A well-known recreational puzzle concerns the bridges of Königsberg. It is claimed that in the early eighteenth century the citizens of Königsberg used to spend their Sunday afternoons walking around their beautiful city. The city itself consisted of four land areas separated by branches of the river Pregel over which there were seven bridges, as illustrated in Figure 1. The problem that the citizens set themselves was to

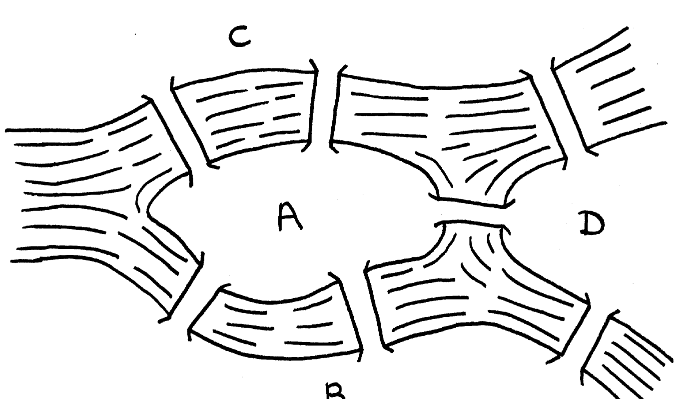

Figure 1. Königsberg

walk around the city, crossing each of the seven bridges exactly once and, if possible, returning to their starting point.

If you look in some books on recreational mathematics, or listen to some graph-theorists who should know better, you will ‘learn’ that Leonhard Euler investigated the Königsberg bridges problem by drawing a graph of the city, as in Figure 2, with a vertex representing each of the four land areas and an edge representing each of the seven bridges. The problem is then to find a trail in this graph that passes along each edge just once.

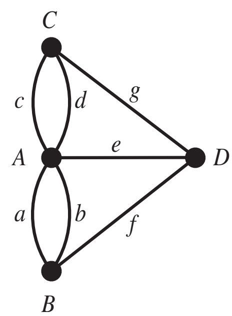

Figure 2. The Königsberg graph

But Euler didn’t draw the graph in Figure 2—graphs of this kind didn’t make their first appearance until the second half of the nineteenth century. So what exactly did Euler do?

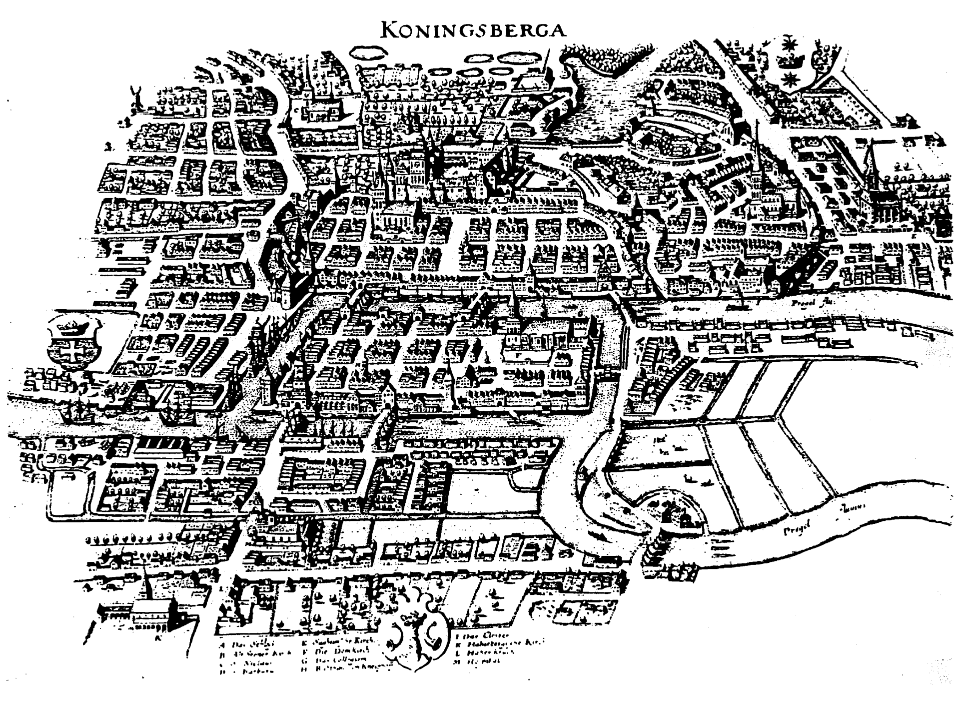

Figure 3. Seventeenth-century Königsberg

The Königsberg bridges problem

In 1254 the Teutonic knights founded the Prussian city of Königsberg (literally, king's mountain). With its strategic position on the river Pregel, it became a trading center and an important medieval city. The river flowed around the island of Kneiphof (literally, pub yard) and divided the city into four regions connected by seven bridges: Blacksmith's bridge, Connecting bridge, High bridge, Green bridge, Honey bridge, Merchant's bridge, and Wooden bridge: Figure 3 shows a seventeenth-century map of the city. Königsberg later became the capital of East Prussia and more recently became the Russian city of Kaliningrad, while the river Pregel was renamed Pregolya.

In 1727 Leonhard Euler began working at the Academy of Sciences in St Petersburg. He presented a paper to his colleagues on 26 August 1735 on the solution of 'a problem relating to the geometry of position': this was the Königsberg bridges problem. He also addressed the generalized problem: given any division of a river into branches and any arrangement of bridges, is there a general method for determining whether such a route exists?

In 1736 Euler wrote up his solution in his celebrated paper in the *Commentarii Academiae Scientiarum Imperialis Petropolitanae* under the title 'Solutio problematis ad geometriam situs pertinentis' [**2**]; Euler's diagram of the Königsberg bridges appears in Figure 4. Although dated 1736, Euler's paper was not actually published until 1741, and was later reprinted in the new edition of the *Commentarii* (*Novi Acta Commentarii* ...) which appeared in 1752.

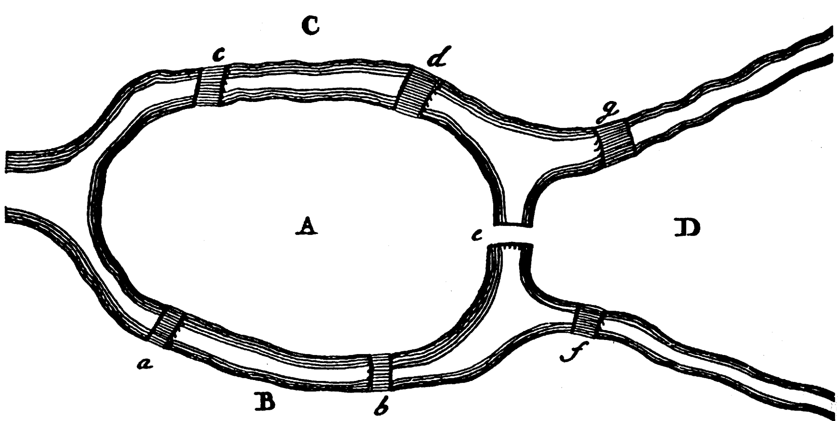

Figure 4. Diagram from Euler's 1736 paper

A full English translation of this paper appears in several places—for example, in [**1**] and [**6**]. The paper begins:

> 1. In addition to that branch of geometry which is concerned with distances, and which has always received the greatest attention, there is another branch, hitherto almost unknown, which Leibniz first mentioned, calling it the geometry of position [*Geometriam situs*]. This branch is concerned only with the determination of position and its properties; it does not involve distances, nor calculations made with them. It has not yet been satisfactorily determined what kinds of problem are relevant to this geometry of position, or what methods should be used in solving them. Hence, when a problem was recently mentioned which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it

was concerned with the geometry of position—especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this problem, as an example of the geometry of position.

2. The problem, which I am told is widely known, is as follows: in Königsberg ...

This reference to Leibniz and the geometry of position dates back to 8 September 1679, when the mathematician and philosopher Gottfried Wilhelm Leibniz wrote to Christiaan Huygens as follows [**5**]:

> I am not content with algebra, in that it yields neither the shortest proofs nor the most beautiful constructions of geometry. Consequently, in view of this, I consider that we need yet another kind of analysis, geometric or linear, which deals directly with position, as algebra deals with magnitudes ...

Leibniz introduced the term *analysis situs* (or *geometria situs*), meaning the analysis of situation or position, to introduce this new area of study. Although it is sometimes claimed that Leibniz had vector analysis in mind when he coined this phrase (see, for example, [**8**] and [**11**]), it was widely interpreted by his eighteenth-century followers as referring to topics that we now consider 'topological'—that is, geometrical in nature, but with no reference to metrical ideas such as distance, length or angle.

Euler's Königsberg letters

It is not known how Euler became aware of the Königsberg bridges problem. However, as we shall see, three letters from the Archive Collection of the Academy of Sciences in St Petersburg [**3**] shed some light on his interest in the problem (see also [**10**]).

Carl Leonhard Gottlieb Ehler was the mayor of Danzig in Prussia (now Gdansk in Poland), some 80 miles west of Königsberg. He corresponded with Euler from 1735 to 1742, acting as intermediary for Heinrich Kühn, a local mathematics professor. Their initial communication has not been recovered, but a letter of 9 March 1736 indicates they had discussed the problem and its relation to the 'calculus of position':

> You would render to me and our friend Kühn a most valuable service, putting us greatly in your debt, most learned Sir, if you would send us the solution, which you know well, to the problem of the seven Königsberg bridges, together with a proof. It would prove to be an outstanding example of the calculus of position [*Calculi Situs*], worthy of your great genius. I have added a sketch of the said bridges ...

Euler replied to Ehler on 3 April 1736, outlining more clearly his own attitude to the problem and its solution:

> ... Thus you see, most noble Sir, how this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are solved more quickly by mathematicians than by others. In the meantime, most noble Sir, you have assigned this question to the geometry of position, but I am ignorant as to what this new discipline involves, and as to which types of problem Leibniz and Wolff expected to see expressed in this way ...

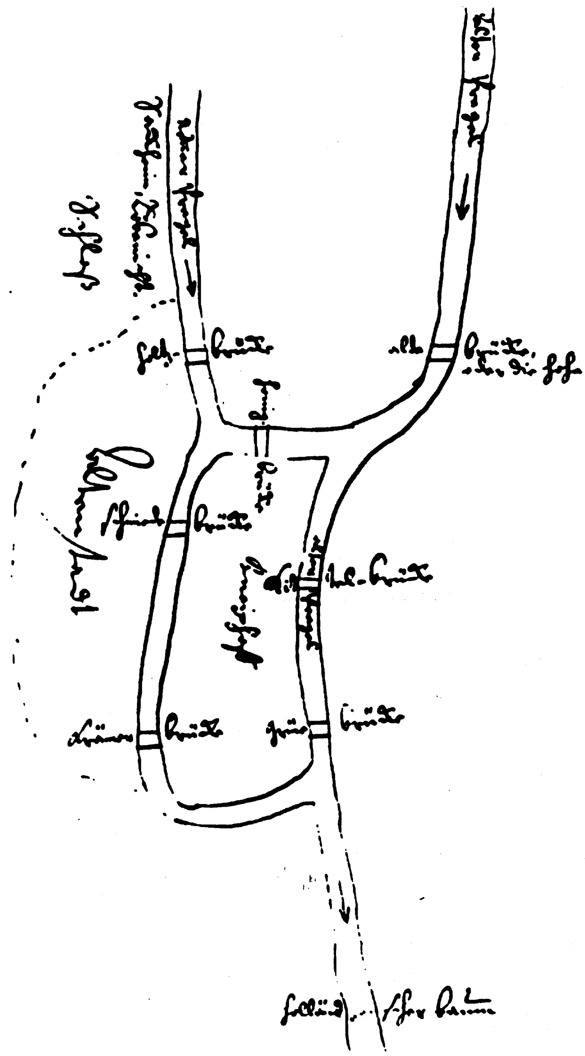

Figure 5. Ehler's letter to Euler

Around the same time, on 13 March 1736, Euler wrote to Giovanni Marinoni, an Italian mathematician and engineer who lived in Vienna and was Court Astronomer in the court of Kaiser Leopold I. He introduced the problem as follows (see Figure 6):

> A problem was posed to me about an island in the city of Königsberg, surrounded by a river spanned by seven bridges, and I was asked whether someone could traverse the separate bridges in a connected walk in such a way that each bridge is crossed only once. I was informed that hitherto no-one had demonstrated the possibility of doing this, or shown that it is impossible. This question is so banal, but seemed to me worthy of attention in that geometry, nor algebra, nor even the art of counting was sufficient to solve it. In view of this, it occurred to me to wonder whether it belonged to the geometry of position [*geometriam Situs*], which Leibniz had once so much longed for. And so, after some deliberation, I obtained a simple, yet completely established, rule with whose help one can immediately decide for all examples of this kind, with any number of bridges in any arrangement, whether such a round trip is possible, or not . . .

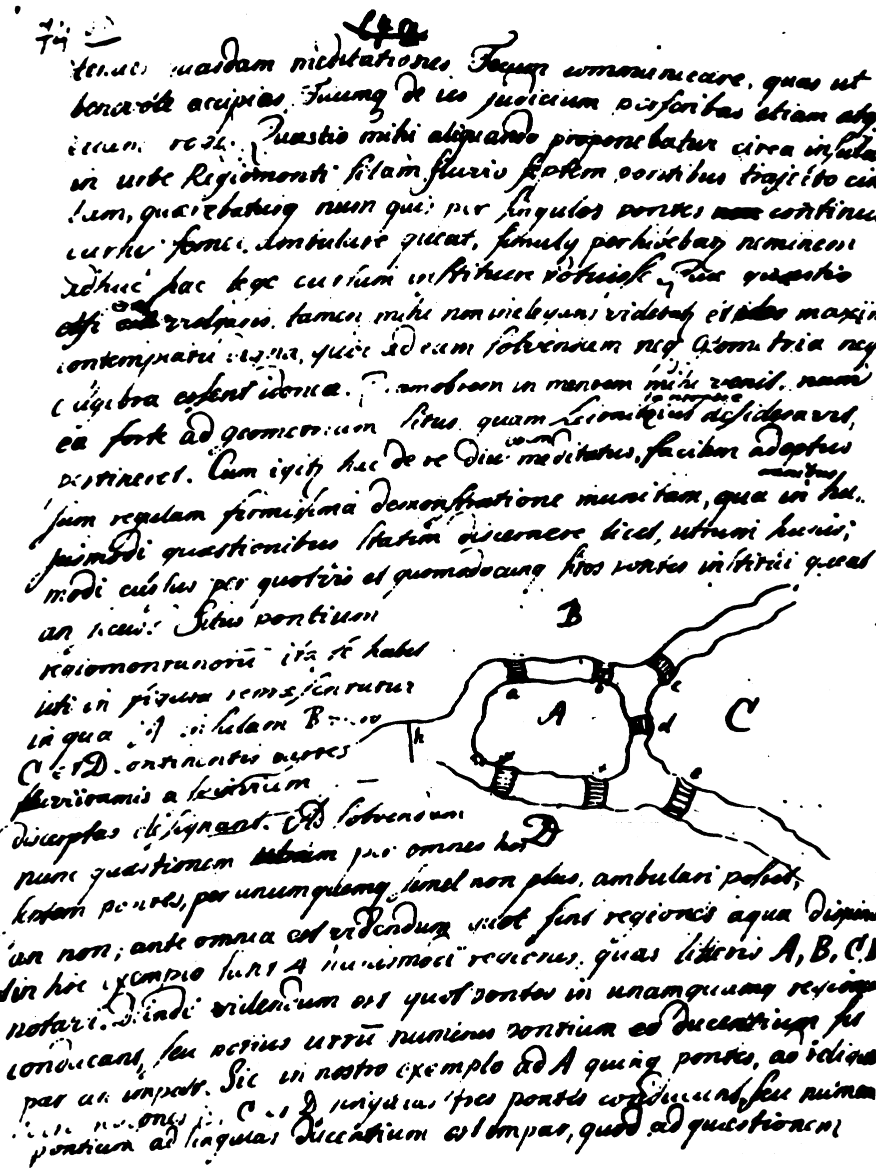

Figure 6. Euler's letter to Marinoni

Euler's 1736 paper

Euler's paper is divided into twenty-one numbered paragraphs, of which the first ascribes the problem to the geometry of position as we saw above, the next eight are devoted to the solution of the Königsberg bridges problem itself, and the remainder are concerned with the general problem. More specifically, paragraphs 2–21 deal with the following topics (see also [**12**]):

Paragraph 2. Euler described the problem of the Königsberg bridges and its generalization: 'whatever be the arrangement and division of the river into branches, and however many bridges there be, can one find out whether or not it is possible to cross each bridge exactly once?'

Paragraph 3. In principle, the original problem could be solved exhaustively by checking all possible paths, but Euler dismissed this as 'laborious' and impossible for configurations with more bridges.

Paragraphs 4–7. The first simplification is to record paths by the land regions rather than bridges. Using the notation in Figure 4, going south from Kneiphof would be notated *AB* whether one used the Green Bridge or the Blacksmith's Bridge. The final path notation will need to include an adjacent *A* and *B* twice; the particular assignment of bridges a and b is irrelevant. A path signified by n letters corresponds to crossing $n-1$ bridges, so a solution to the Königsberg problem requires an eight-letter path with two adjacent A/B pairs, two adjacent A/C pairs, one adjacent A/D pair, etc.

Paragraph 8. What is the relation between the number of bridges connecting a land mass and the number of times the corresponding letter occurs in the path? Euler developed the answer from a simpler example (see Figure 7). If there is an odd number k of bridges, then the letter must appear $(k+1)/2$ times.

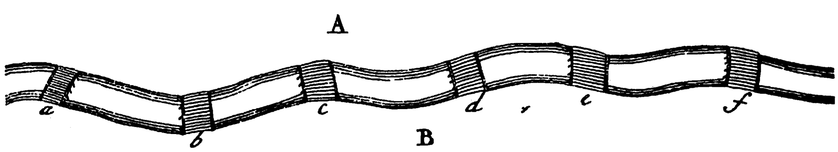

Figure 7. A simple case

Paragraph 9. This is enough to establish the impossibility of the desired Königsberg tour. Since Kneiphof is connected by five bridges, the path must contain three As. Similarly, there must be two Bs, two Cs, and two Ds. In *paragraph 14*, Euler records these data in a table.

region	A	B	C	D
bridges	5	3	3	3
frequency	3	2	2	2

Summing the final row gives nine required letters, but a path using each of the seven bridges exactly once can have only eight letters. Thus there can be no Königsberg tour.

Paragraphs 10–12. Euler continued his analysis from *paragraph 8:* if there is an even number k of bridges connecting a land mass, then the corresponding letter appears $k/2+1$ times if the path begins in that region, and $k/2$ times otherwise.

Paragraphs 13–15. The general problem can now be addressed. To illustrate the method Euler constructed an example with two islands, four rivers, and fifteen bridges (see Figure 8).

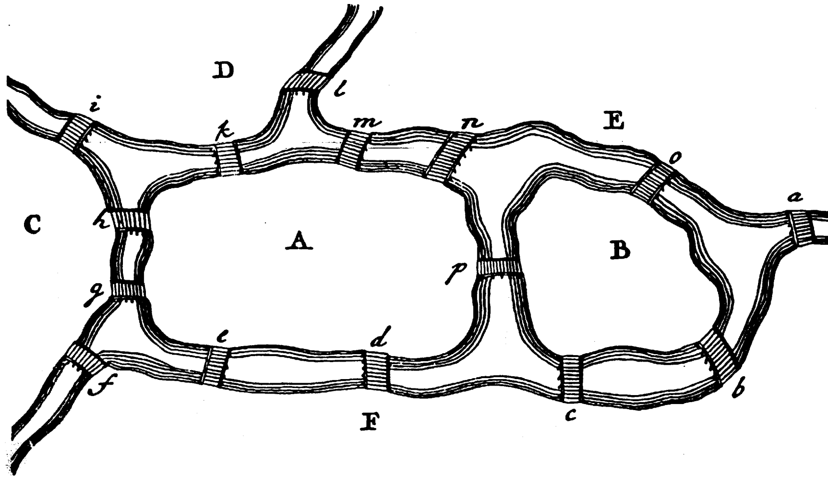

Figure 8. A more complicated example

This system has the following table, where an asterisk indicates a region with an even number of bridges.

region	A^*	B^*	C^*	D	E	F^*
bridges	8	4	4	3	5	6
frequency	4	2	2	2	3	3

The frequencies of the letters in a successful path are determined by the rules for even and odd numbers of bridges, developed above. Since there can be only one initial region, he records $k/2$ for the asterisked regions. If the frequency sum is one less than the required number of letters, there is a path using each bridge exactly once that begins in an asterisked region. If the frequency sum equals the required number of letters, there is a path that begins in an unasterisked region. This latter possibility is the case here: the frequency sum is 16, exactly the number of letters required for a path using 15 bridges. Euler exhibited a particular path, including the bridges:

$$E\,a\,F\,b\,B\,c\,F\,d\,A\,e\,F\,f\,C\,g\,A\,h\,C\,i\,D\,k\,A\,m\,E\,n\,A\,p\,B\,o\,E\,l\,D.$$

Paragraph 16–19. Euler continued with a simpler technique, observing that:

> ... the number of bridges written next to the letters A, B, C, etc. together add up to twice the total number of bridges. The reason for this is that, in the calculation where every bridge leading to a given area is counted, each bridge is counted twice, once for each of the two areas which it joins.

This is the earliest version known of what is now called the *handshaking lemma*. It follows that in the bridge sum, there must be an even number of odd summands.

Paragraph 20. Euler stated his main conclusions:

> If there are more than two areas to which an odd number of bridges lead, then such a journey is impossible.

If, however, the number of bridges is odd for exactly two areas, then the journey is possible if it starts in either of these two areas.

If, finally, there are no areas to which an odd number of bridges lead, then the required journey can be accomplished starting from any area.

Paragraph 21. Euler concluded by saying:

> When it has been determined that such a journey can be made, one still has to find how it should be arranged. For this I use the following rule: let those pairs of bridges which lead from one area to another be mentally removed, thereby considerably reducing the number of bridges; it is then an easy task to construct the required route across the remaining bridges, and the bridges which have been removed will not significantly alter the route found, as will become clear after a little thought. I do not therefore think it worthwhile to give any further details concerning the finding of the routes.

Note that this final paragraph does not prove the existence of a journey when one is possible, apparently because Euler did not consider it necessary. So Euler provided a rigorous proof only for the first of the three conclusions. The first satisfactory proof of the other two results did not appear until 1871, in a posthumous paper by Carl Hierholzer (see [**1**] and [**4**]).

The modern solution

The approach mentioned in the first section developed through diagram-tracing puzzles discussed by Louis Poinsot [**7**] and others in the early-nineteenth century. The object is to determine whether a figure can be drawn with a single stroke of the pen in such a way that no edge is repeated. Considering the figure to be drawn as a graph, the general conditions in *Paragraph 20* take the following form:

> If there are more than two vertices of odd degree, then such a drawing is impossible.
>
> If, however, exactly two vertices have odd degree, then the drawing is possible if it starts with either of these two vertices.
>
> If, finally, there are no vertices of odd degree, then the required drawing can be accomplished starting from any vertex.

So the 4-vertex graph shown in Figure 2, with one vertex of degree 5 and three vertices of degree 3, cannot be drawn with a single stroke of the pen so that no edge is repeated. In contemporary terminology, we say that this graph is not Eulerian. The arrangement of bridges in Figure 8 can be similarly represented by the graph in Figure 9, with six vertices and fifteen edges. Exactly two vertices (E and D) have odd degree, so there is a drawing that starts at E and ends at D, as we saw above. This is sometimes called an Eulerian trail.

However, it was some time until the connection was made between Euler's work and diagram-tracing puzzles. The 'Königsberg graph' of Figure 2 made its first appearance in W. W. Rouse Ball's *Mathematical Recreations and Problems of Past and Present Times* [**9**] in 1892.

Background information, including English translations of the papers of Euler [**2**] and Hierholzer [**4**], can be found in [**1**]; an English translation of Euler's paper also appears in [**6**].

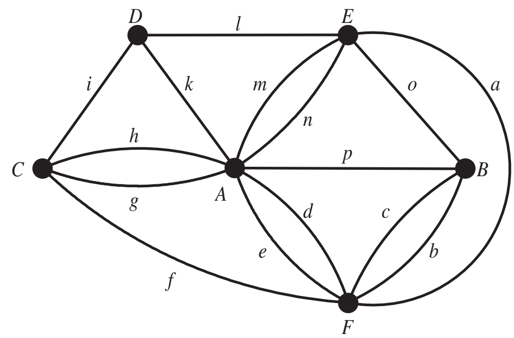

Figure 9. The graph of the bridges in Figure 8

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If I feel unhappy, I do mathematics to become happy. If I am happy, I do mathematics to keep happy. ——Alfréd Rényi

(Quoted in P. Turán, "The Work of Alfréd Rényi," *Matematikai Lapok* **21** (1970) 199–210)